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2010 J. Phys. A: Math. Theor. 43 145207

(http://iopscience.iop.org/1751-8121/43/14/145207)

J. Phys. A: Math. Theor. **43** (2010) 145207 (15pp) doi:10.1088/1751-8113/43/14/145207

Spectral analysis of a q-difference operator

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Received 19 November 2009, in final form 22 February 2010
Published 18 March 2010
Online at stacks.iop.org/JPhysA/43/145207

Abstract
For a number q bigger than 1, we consider a q-difference version of a second-order singular differential operator which depends on a real parameter. We give three exact parameter intervals in which the operator is semibounded from above, not semibounded, and semibounded from below, respectively. We also provide two exact parameter sets in which the operator is symmetric and self-adjoint, respectively. Our model exhibits a more complex behavior than in the classical continuous case but reduces to it when q approaches 1.

PACS numbers: 02.30.Lt, 02.30.Sa, 02.30.Tb, 02.20.Uw
Mathematics Subject Classification: 47B39, 47B25, 47B37, 39A12, 39A13, 34N05

1. Introduction

Let $\mathcal{H}$ be a densely defined symmetric operator on a Hilbert space $\mathfrak{H}$ with domain $\mathcal{D}(\mathcal{H})$. Let U be a unitary operator and let $a \in (0, \infty)\backslash\{1\}$. The operator $\mathcal{H}$ is said to be (a, U)-invariant [2, 3, 9] if

$$U\mathcal{D}(\mathcal{H}) = \mathcal{D}(\mathcal{H})$$

and for any $f \in \mathcal{D}(\mathcal{H})$,

$$U\mathcal{H}f = a\mathcal{H}Uf.$$

From the definition of an (a, U)-invariant operator, it follows that such an operator is either semibounded by zero (i.e., $(\mathcal{H}f, f) \geqslant 0$ or $(\mathcal{H}f, f) \leqslant 0$ for all $f \in \mathcal{D}(\mathcal{H})$) or not semibounded at all. It is well known [7] that any semibounded Hermitian operator always admits semibounded self-adjoint extensions H with the same bound. If such an extension is not unique, then the set of all such extensions contains two extreme elements, the Friedrichs extension H_F and the Kreĭn extension H_K. In [2, 3, 9] it was proved that a semibounded (a, U)-invariant operator always admits semibounded (a, U)-invariant self-adjoint extensions. In particular, the extreme extensions, H_F and H_K, are always (a, U)-invariant. In [2, 3] it was also proved that if the index of defect of the operator $\mathcal{H}$ is $(1, 1)$, then H_F and H_K are the only (a, U)-invariant self-adjoint extensions of $\mathcal{H}$.

In [2] it was also shown that any positive operator $\mathcal{H}$ with the index of defect $(1, 1)$ which is (a_s, U_s)-invariant, where $\{U_s : s \in \mathbb{R}\}$ is a continuous group, is unitarily equivalent to the operator acting on $L^2(0, \infty)$ defined by means of the differential expression

$$(\mathcal{H}_0 x)(t) := -\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \frac{\alpha}{t^2} x(t) \tag{1.1}$$

for some $\alpha \in [-1/4, 3/4)$. The operator $\mathcal{H}_0$ is defined on smooth functions with compact support within $(0, \infty)$. It is well known that for $\alpha \geqslant 3/4$, the operator $\mathcal{H}_0$ is essentially self-adjoint (see [10, chapter VI, section 21, on pages 284 and 285]) and positive (see [15, example 6.A.6 on page 107]), i.e., its closure H is a positive self-adjoint operator. The spectrum of H is simple and absolutely continuous. For $-1/4 \leqslant \alpha < 3/4$, the operator $\mathcal{H}$, the closure of $\mathcal{H}_0$, is a positive symmetric operator (i.e., $(\mathcal{H}x, x) \geqslant 0$ for any function x from the domain of $\mathcal{H}$) with defect numbers $(1, 1)$. For $\alpha = -1/4$, the operator $\mathcal{H}$ has a unique positive extension, and for $-1/4 < \alpha < 3/4$, it has a positive extension that is not unique. The spectrum of any positive self-adjoint extension of $\mathcal{H}$ is continuous. For $\alpha < -1/4$, the operator $\mathcal{H}$ is symmetric, but not semibounded, with defect numbers $(1, 1)$. For such values of α, the spectrum of any self-adjoint extension H of $\mathcal{H}$ has a continuous part, filling the positive semi-axis, and a discrete part, located on the negative semi-axis.

In the present paper, we start to investigate a difference version of the differential operator defined by (1.1), which is (q^2, U)-invariant, where $q > 1$ and U is defined below in theorem 2.4. We show that some properties of the constructed operator closely resemble the properties of the differential operator. The results presented in this paper can be useful for numerical calculations related to the differential operator defined by (1.1) and for the study of one-dimensional fractal structures [8, 11, 14]. Moreover, these results are important from the point of view of dynamic equations on time scales [4], unifying continuous and discrete calculus, as well as the related area of quantum calculus [5, 6].

The setup of this paper is as follows. The next section features some introduction and first results on the q-difference operator under consideration. In section 3, we present some auxiliary convergence results that are needed in the proof of our main result in section 4. The paper concludes with a summary of our findings in section 5.

2. The q-difference operator

Let S be the linear space of all sequences $x = \{x_n\}_{n\in\mathbb{Z}}$ with complex entries. A discrete version of the operator $\mathcal{H}_0$, discussed in the present paper, is constructed in the following way. Select a number $q > 1$ and consider points $t_n = q^n$, $n \in \mathbb{Z}$, as points of discretization. The first and the second derivatives of a function x, defined on $(0, \infty)$ such that $x = \{x_n\}_{n\in\mathbb{Z}} = \{x(q^n)\}_{n\in\mathbb{Z}}$, are replaced by the expressions

$$(\mathrm{D}_q x)_n = \frac{x_{n+1} - x_n}{q^{n+1} - q^n} = \frac{x_{n+1} - x_n}{q^n(q - 1)}$$

and

$$\left(\mathrm{D}_q^2 x\right)_{n-1} = \frac{(\mathrm{D}_q x)_n - (\mathrm{D}_q x)_{n-1}}{q^n - q^{n-1}} = \frac{x_{n+1} - (1+q)x_n + q x_{n-1}}{q^{2n-1}(q - 1)^2},$$

respectively. Let

$$\alpha \in \mathbb{R} \qquad \text{and} \qquad \beta = 1 + q + (q-1)^2\alpha. \tag{2.1}$$

We will study the discrete version of the differential operator (1.1), namely the mapping $\mathcal{L}$ defined for any $x \in S$ by the formula

$$\begin{aligned}(\mathcal{L}x)_n &= -\left(\mathrm{D}_q^2 x\right)_{n-1} + \frac{\alpha}{q^{n-1}q^n} x_n \\ &= -\frac{x_{n+1} - (1+q)x_n + qx_{n-1}}{q^{2n-1}(q-1)^2} + \frac{\alpha}{q^{2n-1}} x_n \\ &= -\frac{x_{n+1} - \beta x_n + qx_{n-1}}{q^{2n-1}(q-1)^2}.\end{aligned} \tag{2.2}$$

Let $l_0^2(\mathbb{Z}; q)$ be the linear subset of S consisting of sequences with finite support. Evidently, the transformation $\mathcal{L}$ maps $l_0^2(\mathbb{Z}; q)$ into itself. Instead of the space $L^2(\mathbb{R}_+)$, we consider the linear space $l^2(\mathbb{Z}; q)$ of all sequences $\{x_n\}_{n\in\mathbb{Z}}$ with complex entries such that

$$\sum_{n=-\infty}^{\infty} q^n |x_n|^2 < \infty.$$

It is clear that $l_0^2(\mathbb{Z}; q) \subset l^2(\mathbb{Z}; q)$. With the inner product defined by

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} (q^n - q^{n-1}) x_n \overline{y_n} = (q-1) \sum_{n=-\infty}^{\infty} q^{n-1} x_n \overline{y_n}, \tag{2.3}$$

the space $l^2(\mathbb{Z}; q)$ becomes a Hilbert space and $l_0^2(\mathbb{Z}; q)$ is dense in $l^2(\mathbb{Z}; q)$. Let L_0 be the linear operator defined on $l_0^2(\mathbb{Z}; q)$ by the mapping $\mathcal{L}$. Define

$$\alpha_- := -\frac{1}{(\sqrt{q}-1)^2} \qquad \text{and} \qquad \alpha_+ := -\frac{1}{(\sqrt{q}+1)^2}. \tag{2.4}$$

Note that the definition (2.1) of β implies

$$\beta - 2\sqrt{q} = (\sqrt{q}-1)^2 + (q-1)^2\alpha = (q-1)^2(\alpha - \alpha_+) \tag{2.5}$$

and

$$\beta + 2\sqrt{q} = (\sqrt{q}+1)^2 + (q-1)^2\alpha = (q-1)^2(\alpha - \alpha_-), \tag{2.6}$$

and thus it is obvious that β is an increasing linear function of α such that $\alpha \in (\alpha_-, \alpha_+)$ if and only if $\beta \in (-2\sqrt{q}, 2\sqrt{q})$.

We now show that the operator L_0 is symmetric, positive when $\alpha \geqslant \alpha_+$, negative when $\alpha \leqslant \alpha_-$ and not semibounded when $\alpha_- < \alpha < \alpha_+$. The first three statements are the contents of the following theorem 2.1, while the last statement is implied by the subsequent remark 2.2.

Theorem 2.1. *For any $x, y \in l_0^2(\mathbb{Z}; q)$, the following relations hold:*

$$\langle L_0x, y\rangle = \langle x, L_0y\rangle, \qquad \langle L_0x, x\rangle \geqslant 0 \quad \text{for} \quad \alpha \geqslant \alpha_+, \qquad \langle L_0x, x\rangle \leqslant 0 \quad \text{for} \quad \alpha \leqslant \alpha_-.$$

Proof. Suppose that $x, y \in l_0^2(\mathbb{Z}; q)$. Then (note that all sums in the following calculation are in fact finite)

$$\begin{aligned}\langle L_0x, y\rangle &= -\sum_{n=-\infty}^{\infty} \frac{x_{n+1} - \beta x_n + qx_{n-1}}{q^n(q-1)} \overline{y_n} \\ &= -\left[\sum_{n=-\infty}^{\infty} \frac{x_n \overline{y_{n-1}}}{q^{n-1}(q-1)} - \sum_{n=-\infty}^{\infty} \frac{\beta x_n \overline{y_n}}{q^n(q-1)} + \sum_{n=-\infty}^{\infty} \frac{q x_n \overline{y_{n+1}}}{q^{n+1}(q-1)}\right] \\ &= -\sum_{n=-\infty}^{\infty} x_n \overline{\frac{y_{n+1} - \beta y_n + qy_{n-1}}{q^n(q-1)}} = \langle x, L_0y\rangle.\end{aligned}$$

This completes the proof of the first statement. In order to prove the second and the third statements, observe that $\langle L_0 x, x\rangle$ can be written in the form

$$\begin{aligned}\langle L_0 x, x\rangle &= -\sum_{n=-\infty}^{\infty} \frac{x_{n+1} - \beta x_n + q x_{n-1}}{q^n(q-1)}\overline{x_n} \\ &= -\left[\sum_{n=-\infty}^{\infty} \frac{x_{n+1}\overline{x_n}}{q^n(q-1)} - \sum_{n=-\infty}^{\infty} \frac{\beta |x_n|^2}{q^n(q-1)} + \sum_{n=-\infty}^{\infty} \frac{q x_n \overline{x_{n+1}}}{q^{n+1}(q-1)}\right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\beta |x_n|^2}{q^n(q-1)} - 2\,\mathrm{Re} \sum_{n=-\infty}^{\infty} \frac{x_{n+1}\overline{x_n}}{q^n(q-1)}. \end{aligned} \tag{2.7}$$

We rewrite this equation and use the Cauchy–Schwarz inequality to get

$$\begin{aligned}\left|\langle L_0 x, x\rangle - \beta \sum_{n=-\infty}^{\infty} \frac{|x_n|^2}{q^n(q-1)}\right| &= 2\left|\mathrm{Re} \sum_{n=-\infty}^{\infty} \frac{x_{n+1}}{q^{n/2}\sqrt{q-1}} \frac{\overline{x_n}}{q^{n/2}\sqrt{q-1}}\right| \\ &\leqslant 2\sqrt{\sum_{n=-\infty}^{\infty} \frac{|x_{n+1}|^2}{q^n(q-1)} \sum_{n=-\infty}^{\infty} \frac{|x_n|^2}{q^n(q-1)}} \\ &= 2\sqrt{q} \sum_{n=-\infty}^{\infty} \frac{|x_n|^2}{q^n(q-1)} \end{aligned}$$

so that

$$(\beta - 2\sqrt{q}) \sum_{n=-\infty}^{\infty} \frac{|x_n|^2}{q^n(q-1)} \leqslant \langle L_0 x, x\rangle \leqslant (\beta + 2\sqrt{q}) \sum_{n=-\infty}^{\infty} \frac{|x_n|^2}{q^n(q-1)}.$$

Therefore, by (2.6) the operator L_0 is negative provided $\alpha \leqslant \alpha_-$, and by (2.5) is positive provided $\alpha \geqslant \alpha_+$. This concludes the proof. □

Remark 2.2. The constants α_+ and α_- are exact, i.e., for all $K > 0$ and

- for all $\alpha < \alpha_+$, there exists $x^+ \in l_0^2(\mathbb{Z}; q)$ with $\langle L_0 x^+, x^+\rangle < -K$;
- for all $\alpha > \alpha_-$, there exists $x^- \in l_0^2(\mathbb{Z}; q)$ with $\langle L_0 x^-, x^-\rangle > K$.

Proof. Let $K > 0$. First suppose $\alpha < \alpha_+$. Now pick any integer

$$N > \frac{(q-1)K + 2\sqrt{q}}{(q-1)^2(\alpha_+ - \alpha)} \qquad \text{and define} \quad x_n^+ = \begin{cases} q^{n/2} & \text{if} \quad 1 \leqslant n \leqslant N \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^+ \in l_0^2(\mathbb{Z}; q)$ and

$$\langle L_0 x^+, x^+\rangle \overset{(2.7)}{=} \frac{\beta N}{q-1} - \frac{2\sqrt{q}(N-1)}{q-1} \overset{(2.5)}{=} \frac{(q-1)^2(\alpha - \alpha_+)N}{q-1} + \frac{2\sqrt{q}}{q-1} < -K.$$

Now suppose $\alpha > \alpha_-$. Pick any integer

$$N > \frac{(q-1)K + 2\sqrt{q}}{(q-1)^2(\alpha - \alpha_-)} \qquad \text{and define} \quad x_n^- = \begin{cases} (-1)^n q^{n/2} & \text{if} \quad 1 \leqslant n \leqslant N \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^- \in l_0^2(\mathbb{Z}; q)$ and

$$\langle L_0 x^-, x^-\rangle \overset{(2.7)}{=} \frac{\beta N}{q-1} + \frac{2\sqrt{q}(N-1)}{q-1} \overset{(2.6)}{=} \frac{(q-1)^2(\alpha - \alpha_-)N}{q-1} - \frac{2\sqrt{q}}{q-1} > K.$$

Hence both α_+ and α_- are exact. □

By theorem 2.1, the operator L_0 is symmetric, and therefore it admits closure. Let L be the closure of L_0, i.e., $L = \overline{L_0}$. L is a densely defined symmetric (possibly, self-adjoint) operator. For $\alpha \geqslant \alpha_+$, the operator L is positive, i.e., $\langle Lx, x\rangle \geqslant 0$ for any $x \in \mathcal{D}(L)$, where $\mathcal{D}(L)$ denotes the domain of L, and for $\alpha \leqslant \alpha_-$, the operator L is negative. It is well known (see, e.g., [1]) that $L^* = L_0^*$. Next we give an explicit description of the operator L^*.

Theorem 2.3. *Let $\mathcal{L} : S \to S$ be the mapping defined by (2.2). Then, the operator L^* is well defined on the linear set*

$$D = \left\{x \in l^2(\mathbb{Z}; q) : \ \mathcal{L}x \in l^2(\mathbb{Z}; q)\right\},$$

and we have $L^*x = \mathcal{L}x$ for any $x \in D$.

Proof. Suppose that the vectors $x, v \in l^2(\mathbb{Z}; q)$ satisfy the equation

$$\langle L_0 y, x\rangle = \langle y, v\rangle \qquad \text{for every} \quad y \in l_0^2(\mathbb{Z}; q). \tag{2.8}$$

Then, for every $k \in \mathbb{Z}$, we can choose $y = y^{(k)} \in l_0^2(\mathbb{Z}; q)$ defined by the formula $y_n^{(k)} = \delta_{kn}$ for any $n \in \mathbb{Z}$, where $\delta_{kn} = 1$ if $n = k$ and $\delta_{kn} = 0$ if $n \neq k$. Since the operator L_0 is defined on $l_0^2(\mathbb{Z}; q)$ by the linear mapping $\mathcal{L}$, in view of (2.2) and (2.3), the equation $\langle L_0 y^{(k)}, x\rangle = \langle y^{(k)}, v\rangle$ implies $v_k = (\mathcal{L}x)_k$ for every $k \in \mathbb{Z}$, that is, $v = \mathcal{L}x$. Hence, L_0^* has domain $\mathcal{D}(L_0^*) \subset D$.

Consider the operator F defined on D by the formula $Fx = \mathcal{L}x$. As in the proof of theorem 2.1, for every $y \in l_0^2(\mathbb{Z}; q)$ and $x \in D$, we obtain $\langle L_0 y, x\rangle = \langle y, Fx\rangle$. Therefore, $\mathcal{D}(L_0^*) = D$ and $L_0^* = F$, which, in view of $L^* = L_0^*$, implies $L^* = F$. The proof is now complete. □

Theorem 2.4. *There exists a unitary U such that L is (q^2, U)-invariant.*

Proof. Denote by U the operator on $l^2(\mathbb{Z}; q)$ defined by

$$(Ux)_n = \frac{1}{\sqrt{q}} x_{n-1}. \tag{2.9}$$

Then

$$\|Ux\|^2 = (q-1) \sum_{n=-\infty}^{\infty} q^{n-1} \frac{1}{q} |x_{n-1}|^2 = \|x\|^2 .$$

It is clear that $Ul^2(\mathbb{Z}; q) = l^2(\mathbb{Z}; q)$. Therefore, the operator U is unitary. It is easily seen that

$$(U^*x)_n = \sqrt{q} x_{n+1}. \tag{2.10}$$

Direct calculations show that for any $x \in l_0(\mathbb{Z}; q)$,

$$UL_0 x = q^2 L_0 U x,$$

i.e., the operator L_0 is (q^2, U)-invariant. The same is true for the operator L, the closure of L_0 (see [2, 3, 9]). □

3. Some auxiliary convergence results

For the proof of our main results presented in section 4 below, we will need to know if the equation

$$(\mathcal{L}x)_n = z x_n, \qquad n \in \mathbb{Z}, \tag{3.1}$$

has nontrivial solutions $x \in l^2(\mathbb{Z}; q)$ for certain complex numbers z. In order to discuss the convergence of

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 \tag{3.2}$$

for solutions x of (3.1), we employ the following theorem of Perron. This theorem was proved first by Oskar Perron in 1910 in [12, Fundamentalsatz on page 19] and eleven years later again by the same author in [13, Satz 3 on page 14], this time using a much simpler proof. We formulate Perron's result only in the special situation in which it will be employed in the following.

Theorem 3.1 (Perron's theorem)**.** *Consider the second-order linear difference equation*

$$w_{n+1} + (b + b_n)w_n + cw_{n-1} = 0 \qquad \text{for} \quad n \in \mathbb{Z}, \tag{3.3}$$

where

$$b \in \mathbb{C}, \qquad c \in \mathbb{C}\backslash\{0\}, \qquad b_n \in \mathbb{C} \text{ for } n \in \mathbb{Z} \qquad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ *be the solutions of*

$$\lambda^2 + b\lambda + c = 0. \tag{3.4}$$

Then (3.3) has two linearly independent solutions $w^{(1)}$ *and* $w^{(2)}$ *such that*

$$\limsup_{n\to\infty} \sqrt[n]{\left|w_n^{(1)}\right|} = |\lambda_1| \qquad \text{and} \qquad \limsup_{n\to\infty} \sqrt[n]{\left|w_n^{(2)}\right|} = |\lambda_2|.$$

Observe that Perron's theorem does not require any restrictions on the characteristic roots λ_1 and λ_2, and this is essential for the proof of theorem 3.2 below. Define

$$\alpha_{--} := -\frac{\sqrt{q} - 1 + \frac{1}{\sqrt{q}}}{(\sqrt{q} - 1)^2} \qquad \text{and} \qquad \alpha_{++} := \frac{\sqrt{q} + 1 + \frac{1}{\sqrt{q}}}{(\sqrt{q} + 1)^2}. \tag{3.5}$$

Note that definition (2.1) of β implies

$$\beta - \frac{q^2 + 1}{\sqrt{q}} = 1 + q - q\sqrt{q} - \frac{1}{\sqrt{q}} + (q - 1)^2\alpha = (q - 1)^2(\alpha - \alpha_{++}) \tag{3.6}$$

and

$$\beta + \frac{q^2 + 1}{\sqrt{q}} = 1 + q + q\sqrt{q} + \frac{1}{\sqrt{q}} + (q - 1)^2\alpha = (q - 1)^2(\alpha - \alpha_{--}). \tag{3.7}$$

Theorem 3.2. *Assume* $\alpha \in (\alpha_{--}, \alpha_{++})$. *Let* $z \in \mathbb{C}$ *be arbitrary. Then every solution* $x \in S$ *of (3.1) satisfies*

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 < \infty.$$

Proof. Note that x solves (3.1) if and only if

$$x_{n+1} - (\beta - zq^{2n-1}(q - 1)^2)x_n + qx_{n-1} = 0 \qquad \text{for all} \quad n \in \mathbb{Z},$$

i.e.,

$$x_{-n+1} - (\beta - zq^{-2n-1}(q - 1)^2)x_{-n} + qx_{-n-1} = 0 \qquad \text{for all} \quad n \in \mathbb{Z},$$

i.e., writing $w_n = x_{-n}$,

$$w_{n-1} - (\beta - zq^{-2n-1}(q-1)^2)w_n + qw_{n+1} = 0 \qquad \text{for all} \quad n \in \mathbb{Z},$$

i.e.,

$$w_{n+1} + (b + b_n)w_n + cw_{n-1} = 0 \qquad \text{for all} \quad n \in \mathbb{Z}, \tag{3.8}$$

where

$$b = -\frac{\beta}{q}, \qquad c = \frac{1}{q} \neq 0 \qquad \text{and} \qquad b_n = zq^{-2n-2}(q-1)^2 \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus, we may apply Perron's theorem (theorem 3.1) to conclude that there exist two linearly independent solutions $w^{(1)}$ and $w^{(2)}$ of (3.8) satisfying

$$\limsup_{n\to\infty} \sqrt[n]{\left|w_n^{(1)}\right|} = |\lambda_1| \qquad \text{and} \qquad \limsup_{n\to\infty} \sqrt[n]{\left|w_n^{(2)}\right|} = |\lambda_2|,$$

where λ_1 and λ_2 are solutions of (3.4), i.e.,

$$\lambda_1 = \frac{\beta - \sqrt{\beta^2 - 4q}}{2q} \qquad \text{and} \qquad \lambda_2 = \frac{\beta + \sqrt{\beta^2 - 4q}}{2q}. \tag{3.9}$$

Since $\alpha_{--} < \alpha < \alpha_{++}$, formulas (3.6) and (3.7) yield that

$$|\beta| < \frac{q^2+1}{\sqrt{q}}.$$

Thus,

$$\beta^2 - 4q < \left(\frac{q^2+1}{\sqrt{q}}\right)^2 - 4q = \left(\frac{q^2-1}{\sqrt{q}}\right)^2.$$

Hence, if $\beta^2 - 4q > 0$, then we have

$$|\lambda_i| \leqslant \frac{|\beta| + \sqrt{\beta^2 - 4q}}{2q} < \frac{\frac{q^2+1}{\sqrt{q}} + \frac{q^2-1}{\sqrt{q}}}{2q} = \sqrt{q} \qquad \text{for} \quad i \in \{1, 2\},$$

and, if $\beta^2 - 4q < 0$, then we have

$$|\lambda_i| = \sqrt{\left(\frac{\beta}{2q}\right)^2 + \left(\frac{\sqrt{4q - \beta^2}}{2q}\right)^2} = \frac{1}{\sqrt{q}} < \sqrt{q} \qquad \text{for} \quad i \in \{1, 2\}.$$

Thus, in any case $|\lambda_i| < \sqrt{q}$ for $i \in \{1, 2\}$. Observing that

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 = \sum_{n=1}^{\infty} q^{-n} |w_n|^2, \tag{3.10}$$

we apply the root test to check the convergence of (3.10) by calculating

$$\nu_i := \limsup_{n\to\infty} \sqrt[n]{\left|q^{-n} w_n^{(i)}\right|^2} = \frac{1}{q}\left(\limsup_{n\to\infty} \sqrt[n]{\left|w_n^{(i)}\right|}\right)^2 = \frac{|\lambda_i|^2}{q} < \frac{q}{q} = 1 \qquad \text{for} \quad i \in \{1, 2\}.$$

Hence, there exist two linearly independent solutions of (3.1), each such that the corresponding series (3.2) converges. Then an arbitrary solution of (3.1) may be written as a linear combination of those two solutions, and hence, applying the inequality $|u+v|^2 \leqslant 2(|u|^2+|v|^2)$, the arbitrary solution is also such that (3.2) converges. □

Remark 3.3. Although the following generalization of theorem 3.2 is not needed in our proof of the main result in section 4 below, we mention it here for completeness. Some further

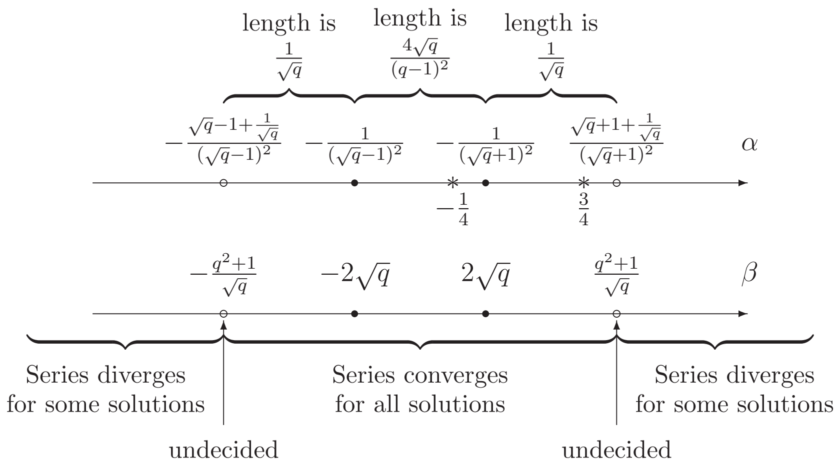

Figure 1. Results of Perron's theorem for series (3.2).

analysis of the two values $|\nu_1|$ and $|\nu_2|$ from the proof of theorem 3.2 yields the results depicted in figure 1.

Now we discuss the remaining values of α.

Theorem 3.4. *Assume $\alpha \in [\alpha_{++}, \infty)$. Let $z \in (-\infty, 0)$ be arbitrary. Then any solution $x \in S$ of (3.1) such that*

$$x_{-1} = 1 \qquad \text{and} \qquad x_{-2} \geqslant \frac{2+q}{\sqrt{q}} \tag{3.11}$$

satisfies

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 = \infty.$$

Proof. Since $\alpha \geqslant \alpha_{++}$, formula (3.6) yields that

$$\beta \geqslant \frac{q^2+1}{\sqrt{q}}.$$

Suppose x solves (3.1) and satisfies (3.11). With $w_n = x_{-n}$ as in the proof of theorem 3.2, (3.1) is equivalently rewritten as (3.8), where

$$b \leqslant -\frac{q+\frac{1}{q}}{\sqrt{q}}, \qquad c = \frac{1}{q} \qquad \text{and} \quad b_n = zq^{-2n-2}(q-1)^2 \nearrow 0 \qquad \text{as} \quad n \to \infty.$$

Thus, w solves (3.8) and satisfies

$$w_1 = 1 \qquad \text{and} \qquad w_2 \geqslant \frac{2+q}{\sqrt{q}}.$$

We first claim that

$$w_n > 0 \qquad \text{and} \qquad \frac{w_{n+1}}{w_n} > \frac{n+1}{n\sqrt{q}} \tag{3.12}$$

holds for all $n \in \mathbb{N}$. This claim will now be proved by induction. First, $w_1 = 1 > 0$ and

$$\frac{w_2}{w_1} = w_2 \geqslant \frac{2+q}{\sqrt{q}} > \frac{2}{\sqrt{q}},$$

so (3.12) holds for $n = 1$. Assume now that (3.12) holds for some $n \in \mathbb{N}$. Then

$$w_{n+1} > \frac{n+1}{n\sqrt{q}} w_n > 0$$

and

$$\begin{aligned}\frac{w_{n+2}}{w_{n+1}} &= -(b+b_{n+1}) - \frac{c}{\frac{w_{n+1}}{w_n}} > \frac{q+\frac{1}{q}}{\sqrt{q}} - \frac{\frac{1}{q}}{\frac{n+1}{n\sqrt{q}}}\\ &= \frac{1}{\sqrt{q}}\left\{q + \frac{1}{q} - \frac{n}{n+1}\right\} = \frac{1}{\sqrt{q}}\left\{\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)^2 + 2 - \frac{n}{n+1}\right\}\\ &> \frac{1}{\sqrt{q}}\left\{2 - \frac{n}{n+1}\right\} = \frac{n+2}{(n+1)\sqrt{q}}\end{aligned}$$

so that (3.12) holds for $n + 1$. Hence, the claim is proved and (3.12) holds for all $n \in \mathbb{N}$. We may therefore note that the sequence of ratios defined by

$$r_n := \frac{w_{n+1}}{w_n} \qquad \text{for all} \quad n \in \mathbb{N}$$

is well defined and satisfies

$$r_{n+1} = -b - b_{n+1} - \frac{c}{r_n} \qquad \text{for all} \quad n \in \mathbb{N}. \tag{3.13}$$

We now claim that there exists $m \in \mathbb{N}$ such that

$$w_{n+1} > \sqrt{q} w_n \qquad \text{for all} \quad n \geqslant m. \tag{3.14}$$

This claim (3.14) yields that $\{q^{-n} w_n^2\}_{n\in\mathbb{N}}$ is positive and eventually strictly increasing and hence does not tend to zero. Thus, (3.14) establishes the divergence of the series (3.10). In order to show (3.14), we first assume

$$r_n \leqslant r_{n+1} \qquad \text{for all} \quad n \in \mathbb{N}. \tag{3.15}$$

From (3.15), we find

$$r_n \geqslant r_1 = \frac{w_2}{w_1} = w_2 \geqslant \frac{q+2}{\sqrt{q}} > \frac{q}{\sqrt{q}} = \sqrt{q} \qquad \text{for all} \quad n \in \mathbb{N}$$

so that (3.14) follows with $m = 1$. Next, if (3.15) does not hold, then

$$r_m > r_{m+1} \qquad \text{for some} \quad m \in \mathbb{N}. \tag{3.16}$$

But applying (3.13) twice now yields

$$r_{m+1} = -b - b_{m+1} - \frac{c}{r_m} > -b - b_{m+2} - \frac{c}{r_{m+1}} = r_{m+2},$$

and hence $r_n > r_{n+1}$ for all $n \geqslant m$. Thus, the sequence $\{r_n\}_{n\in\mathbb{N}}$ is eventually strictly decreasing. Since the sequence $\{r_n\}_{n\in\mathbb{N}}$ is also bounded below by $1/\sqrt{q}$ according to (3.12), we now conclude that this sequence has a limit, say λ, and that

$$r_n > \lambda \geqslant \frac{1}{\sqrt{q}} \qquad \text{for all} \quad n \geqslant m. \tag{3.17}$$

By (3.13), we have $\lambda = -b - c/\lambda$, and so λ solves (3.4). The solutions (3.9) of (3.4) satisfy

$$\lambda_2 \geqslant \frac{\frac{q^2+1}{\sqrt{q}} + \sqrt{\left(\frac{q^2+1}{\sqrt{q}}\right)^2 - 4q}}{2q} = \sqrt{q} \quad \text{and} \quad \lambda_1 = \frac{c}{\lambda_2} = \frac{1}{q\lambda_2} \leqslant \frac{1}{q\sqrt{q}} < \frac{1}{\sqrt{q}}$$

so that $\lambda = \lambda_1$ is impossible due to (3.17). Hence, $\lambda = \lambda_2 \geqslant \sqrt{q}$. Employing (3.17) again, (3.14) follows. □

Theorem 3.5. *Assume $\alpha \in (-\infty, \alpha_{--}]$. Let $z \in (0, \infty)$ be arbitrary. Then any solution $x \in S$ of (3.1) such that*

$$x_{-1} = -1 \qquad \text{and} \qquad x_{-2} \geqslant \frac{2+q}{\sqrt{q}} \tag{3.18}$$

satisfies

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 = \infty.$$

Proof. Since $\alpha \leqslant \alpha_{--}$, formula (3.7) yields that

$$\beta \leqslant -\frac{q^2+1}{\sqrt{q}}.$$

Suppose x solves (3.1) and satisfies (3.18). Define now $y_n = (-1)^n x_n$. Then y solves (3.11) and

$$y_{n+1} - ((-\beta) - (-z)q^{2n-1}(q-1)^2)y_n + q y_{n-1} = 0 \qquad \text{for} \quad n \in \mathbb{Z}.$$

By theorem 3.4,

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 = \sum_{n=-\infty}^{-1} q^n |y_n|^2 = \infty.$$

The proof is complete. □

Remark 3.6. Although the following generalization of theorems 3.4 and 3.5 is not needed in our proof of the main result in section 4 below, we mention it here for completeness. With only slight modifications in the proofs of theorems 3.4 and 3.5, both results remain valid if we replace (3.11) and (3.18) by the condition

$$x_{-1} \neq 0 \qquad \text{and} \qquad \left|\frac{x_{-2}}{x_{-1}}\right| \geqslant \max\left\{\frac{2}{\sqrt{q}}, \sqrt{q}\right\}.$$

4. Index of defect of the operator L

Recall that for an unbounded symmetric operator T on a Hilbert space $\mathfrak{H}$, its defect numbers are defined as dimensions of the kernel of the operator $T^* - zI$, where z is a complex number that belongs to the field of regularity of T. The elements of the kernel of $T^* - zI$ are called defect vectors of T. Defect numbers are constant in each component of the field of regularity. For a symmetric operator, the field of regularity has at most two connected components, and the open sets $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ are subsets of it. The defect numbers $n_\pm$ are defined as $\dim \mathrm{Ker}(T^* - zI)$ for $z \in \mathbb{C}_\pm$, and the ordered pair (n_-, n_+) is called the index of defect of the symmetric operator. The operator is self-adjoint if and only if $n_+ = n_- = 0$.

Since the operator L_0 is defined by a difference expression with real coefficients, the defect numbers of L are equal, i.e., $n_+ = n_-$. If $\alpha \geqslant \alpha_+$ or $\alpha \leqslant \alpha_-$, then it follows from theorem 2.1

that the field of regularity of L is connected because it contains $\mathbb{C} \setminus [0, \infty)$ or $\mathbb{C} \setminus (-\infty, 0]$, respectively.

For the further discussion, we need the following construction. Let $l^2(\mathbb{N}_0; q)$ be the linear space of all sequences $x = \{x_n\}_{n=0}^{\infty}$ which satisfy the condition

$$\sum_{n=0}^{\infty} q^n |x_n|^2 < \infty.$$

The space $l^2(\mathbb{N}_0; q)$ may be regarded as a closed subspace of $l^2(\mathbb{Z}; q)$. Let A be the operator on $l^2(\mathbb{N}_0; q)$ defined by

$$(Ax)_n = \begin{cases} -\dfrac{x_1 - \beta x_0}{q^{-1}(q-1)^2} & \text{if} \quad n = 0 \\ -\dfrac{x_{n+1} - \beta x_n + q x_{n-1}}{q^{2n-1}(q-1)^2} & \text{if} \quad n \in \mathbb{N}. \end{cases} \tag{4.1}$$

Lemma 4.1. *The operator A defined by (4.1) is self-adjoint and compact.*

Proof. Let $e^{(k)} \in l^2(\mathbb{N}_0; q)$ for $k \in \mathbb{N}_0$ be defined by its components

$$e_n^{(k)} = \frac{q^{-(k-1)/2}}{\sqrt{q-1}} \delta_{kn} \quad \text{for } n \in \mathbb{N}_0, \quad \text{where } \delta_{kn} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \in \mathbb{N}_0 \backslash \{k\}. \end{cases} \tag{4.2}$$

It is clear that $\{e^{(k)} : k \in \mathbb{N}_0\}$ is an orthonormal basis of $l^2(\mathbb{N}_0; q)$. One can easily check that

$$\|Ae^{(k)}\|^2 = \frac{1 + \beta^2 q + q^4}{q^{4k-1}(q-1)^4}, \qquad k \in \mathbb{N}, \qquad \text{so that} \quad \sum_{k=0}^{\infty} \|Ae^{(k)}\|^2 < \infty,$$

and hence A is a compact operator of Hilbert–Schmidt class. As in the proof of theorem 2.1, one obtains that $\langle Ax, y \rangle = \langle x, Ay \rangle$. Thus, the operator A, being bounded, is self-adjoint. □

We may use the operator A to prove the following auxiliary result:

Lemma 4.2. *If $A - zI$ is invertible and x is a defect vector of L corresponding to z such that $x_{-1} = 0$, then $x = 0$.*

Proof. Let x be any defect vector of L corresponding to z, i.e., $x \in \mathrm{Ker}(L^* - zI)$. Note that this implies $x \in l^2(\mathbb{Z}; q)$. Define now the vector $x^+ \in l^2(\mathbb{N}_0; q)$ by

$$x_n^+ := x_n \qquad \text{for all} \quad n \in \mathbb{N}_0. \tag{4.3}$$

Since $x_{-1} = 0$, it follows from (4.1) and (3.1) that $Ax^+ = zx^+$. Hence, $(A - zI)x^+ = 0$, which implies $x^+ = 0$ since $A - zI$ is invertible. Since x solves (3.1), this implies $x = 0$. □

The following theorem is the main result of this paper.

Theorem 4.3. *The index of defect of the operator L is*

$$(1, 1) \qquad \text{if} \quad \alpha \in (\alpha_{--}, \alpha_{++})$$

and

$$(0, 0) \qquad \text{if} \quad \alpha \in (-\infty, \alpha_{--}] \cup [\alpha_{++}, \infty).$$

Proof. Let us start with the case when $\alpha_{--} < \alpha < \alpha_{++}$. Consider the operator A defined on $l^2(\mathbb{N}_0; q)$ by (4.1) and (note that A is bounded by lemma 4.1) choose

$$z := \mathrm{i}(1 + \|A\|),$$

which is in the field of regularity of the operator L. Since $|z| > \|A\|$, the operator $A - zI$ has an inverse operator $(A - zI)^{-1}$. Let $e^{(0)}$ be the vector defined by (4.2). Then

$$x^+ := (A - zI)^{-1} e^{(0)} \in l^2(\mathbb{N}_0; q)\backslash\{0\}. \tag{4.4}$$

Let $x \in S$ be the solution of the second-order difference equation (3.1) satisfying the conditions $x_0 = x_0^+$ and $x_1 = x_1^+$. From (4.1) and (3.1), it follows that $x_n = x_n^+$ for every $n \in \mathbb{N}_0$, and hence

$$\sum_{n=0}^{\infty} q^n |x_n|^2 < \infty.$$

On the other hand, theorem 3.2 asserts that

$$\sum_{n=-\infty}^{-1} q^n |x_n|^2 < \infty.$$

Thus, x is a nontrivial (i.e., $x \neq 0$) defect vector of L corresponding to z, which implies $\dim \operatorname{Ker}(L^* - zI) \geqslant 1$. Let $\tilde{x}$ be an arbitrary defect vector corresponding to the same z. Define now $y := \tilde{x}_{-1} x - x_{-1} \tilde{x}$. Then y is a defect vector satisfying $y_{-1} = 0$. By lemma 4.2, $y = 0$. Hence, $\tilde{x}_{-1} x = x_{-1} \tilde{x}$, and therefore (apply lemma 4.2 to the defect vector $x \neq 0$)

$$\tilde{x} = \kappa x, \qquad \text{where} \quad \kappa := \frac{\tilde{x}_{-1}}{x_{-1}}.$$

Thus, we conclude that $\dim \operatorname{Ker}(L^* - zI) = 1$. Hence, the operator L has the index of defect $(1, 1)$.

Now we consider the case $\alpha \geqslant \alpha_{++}$. In this case, theorem 2.1 implies that the field of regularity of L contains $(-\infty, 0)$, and we can choose

$$z := -(\gamma + \|A\|), \qquad \text{where} \quad \gamma := \frac{2q^3\sqrt{q}}{(q-1)^2} > 0. \tag{4.5}$$

It is sufficient to show that the only solution of the difference equation (3.1) in $l^2(\mathbb{Z}; q)$ is the trivial solution $x = 0$. For the sake of contradiction, suppose that there exists a nontrivial defect vector $\tilde{x}$ corresponding to z. Then $\tilde{x}_{-1} \neq 0$ by lemma 4.2 (note that $A - zI$ is invertible because of $|z| > \|A\|$), and now we consider the vector

$$x := \frac{1}{\tilde{x}_{-1}} \tilde{x},$$

which is also a defect vector corresponding to the same z and satisfies $x_{-1} = 1$. Taking into account that x solves (3.1), which is a linear difference equation with real coefficients, we may assume that all entries of x are real, i.e., $x_n \in \mathbb{R}$ for every $n \in \mathbb{Z}$. As in the proof of lemma 4.2, we now define $x^+ \in l^2(\mathbb{N}_0; q)$ by (4.3). Since x is a solution of (3.1), it follows from (4.1) and (3.1) that

$$Ax^+ = zx^+ + \left(\frac{q}{q-1}\right)^{3/2} x_{-1} e^{(0)}, \quad \text{and thus} \quad x^+ = \left(\frac{q}{q-1}\right)^{3/2} (A - zI)^{-1} e^{(0)}.$$

Then, using (4.1) and (3.1) again, we obtain

$$\begin{aligned} |x_0| &= \left| \left(\frac{q}{q-1}\right)^{3/2} \langle (A - zI)^{-1} e^{(0)}, e^{(0)} \rangle \right| \\ &\leqslant \left(\frac{q}{q-1}\right)^{3/2} \|(A - zI)^{-1}\| \|e^{(0)}\|^2 \\ &\leqslant \frac{q^{3/2}}{\gamma (q-1)^{3/2}}, \end{aligned} \tag{4.6}$$

where we have used the inequality

$$\|(A - zI)^{-1}\| \leqslant |z|^{-1}\|(Az^{-1} - I)^{-1}\| \leqslant \frac{1}{|z| - \|A\|}.$$

From (3.1) and (4.6), it follows that

$$\begin{aligned} x_{-2} &= \frac{\beta}{q} - \frac{(q-1)^2}{q^4}z - \frac{x_0}{q} \geqslant \frac{q^2+1}{q\sqrt{q}} + \frac{(q-1)^2}{q^4}\gamma - \frac{\sqrt{q}}{(q-1)^{3/2}\gamma} \\ &\overset{(4.5)}{=} \sqrt{q} + \frac{1}{q\sqrt{q}} + \frac{2}{\sqrt{q}} - \frac{\sqrt{q-1}}{2q^3} = \frac{2+q}{\sqrt{q}} + \frac{2q\sqrt{q} - \sqrt{q-1}}{2q^3} \\ &> \frac{2+q}{\sqrt{q}}. \end{aligned}$$

Thus, (3.11) holds, and theorem 3.4 asserts that $x \notin l^2(\mathbb{Z}; q)$. Therefore, the operator L has the index of defect (0, 0).

Finally, in the case when $\alpha \leqslant \alpha_{--}$, theorem 2.1 implies that the field of regularity of L contains $(0, \infty)$, and we can choose $z := \gamma + \|A\|$ with γ as in (4.5). Following along the lines of the previous part of this proof, we construct first a defect vector x satisfying $x_{-1} = -1$. Inequality (4.6) therefore remains valid. From (3.1) and (4.6), it follows that

$$x_{-2} = -\frac{\beta}{q} + \frac{(q-1)^2}{q^4}z - \frac{x_0}{q} \geqslant \frac{q^2+1}{q\sqrt{q}} + \frac{(q-1)^2}{q^4}\gamma - \frac{\sqrt{q}}{(q-1)^{3/2}\gamma} > \frac{2+q}{\sqrt{q}}.$$

Thus, (3.18) holds, and theorem 3.5 asserts that $x \notin l^2(\mathbb{Z}; q)$. Therefore, the operator L again has the index of defect (0, 0). □

Although the following statement has not been used in the proof of the main theorem, we formulate it here for completeness. It complements lemma 4.2, and its proof is performed using the (q^2, U)-invariance of the operator L.

Lemma 4.4. *If $A - zI$ is invertible and x is a defect vector of L corresponding to z such that $x_n = 0$ for some $n \in \mathbb{Z}$, then $x = 0$.*

Proof. From the fact that the operator L is (q^2, U)-invariant (see theorem 2.4), it follows that the operator L^* is also (q^2, U)-invariant (see [2, 3, 9]), i.e.,

$$UL^* = q^2 L^* U$$

with U as in (2.9). In particular, if $x \in \mathrm{Ker}(L^* - zI)\backslash\{0\}$, then

$$Ux \in \mathrm{Ker}\left(L^* - \frac{z}{q^2}I\right) \backslash \{0\} \qquad \text{and} \qquad U^*x \in \mathrm{Ker}(L^* - zq^2 I)\backslash\{0\}.$$

The numbers z/q^2 and $q^2 z$ belong to the same component of the field of regularity of L as z. It was shown in lemma 4.2 that

$$x_{-1} \neq 0 \quad \text{for any nontrivial defect vector } x.$$

Applying this to the defect vectors Ux and U^*x and using

$$(Ux)_{-1} = \frac{x_{-2}}{\sqrt{q}} \qquad \text{and} \qquad (U^*x)_{-1} = \sqrt{q}x_0$$

(by (2.9) and (2.10), respectively), we deduce that

$$x_0 \neq 0 \qquad \text{and} \qquad x_{-2} \neq 0.$$

We may continue this process and obtain that $x_n \neq 0$ for any $n \in \mathbb{Z}$. □

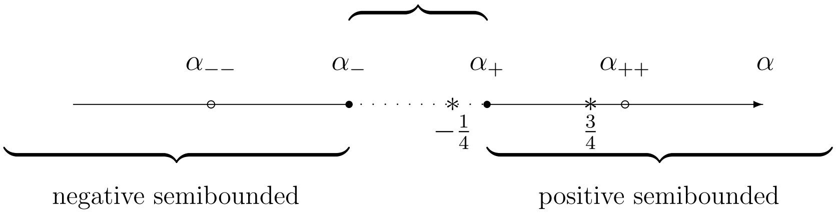

Figure 2. Semiboundedness of the q-difference operator.

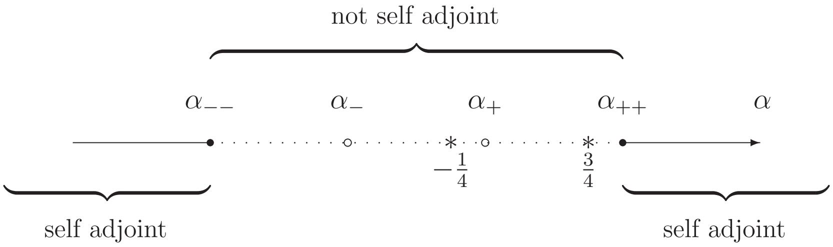

Figure 3. Self-adjointness of the q-difference operator.

5. Conclusions

Let $q > 1$ be fixed and define the Jackson derivative of a function $x : \{q^n : n \in \mathbb{Z}\} \to \mathbb{C}$ by

$$(D_q x)(t) := \frac{x(qt) - x(t)}{(q-1)t} \qquad \text{for} \quad t \in \{q^n : n \in \mathbb{Z}\}.$$

In this paper, we have shown that the q-difference operator defined by

$$-D_q^2 x(t) + \frac{\alpha}{qt^2} x(qt), \qquad \text{where} \quad t \in \{q^n : n \in \mathbb{Z}\},$$

is not semibounded if and only if

$$-\frac{1}{(\sqrt{q}-1)^2} =: \alpha_- < \alpha < \alpha_+ := -\frac{1}{(\sqrt{q}+1)^2}$$

(see figure 2), and that it is not self-adjoint if and only if

$$-\frac{\sqrt{q} - 1 + \frac{1}{\sqrt{q}}}{(\sqrt{q}-1)^2} =: \alpha_{--} < \alpha < \alpha_{++} := \frac{\sqrt{q} + 1 + \frac{1}{\sqrt{q}}}{(\sqrt{q}+1)^2}$$

(see figure 3).

Note that it is well known that both analogs for the corresponding classical differential operator do not contain three intervals each, as above, but simply split the α line into two halves at $-1/4$ for the semiboundedness result and at $3/4$ for the self-adjointness result. One can see nicely how this is resembled when letting $q \to 1$ in our results. While

$$\alpha_+ \to -\tfrac{1}{4} \qquad \text{and} \qquad \alpha_{++} \to \tfrac{3}{4},$$

we see that both

$$\alpha_- \to -\infty \qquad \text{and} \qquad \alpha_{--} \to -\infty$$

as $q \to 1$. Hence, this third barrier in both of the above results is moving to the left when decreasing the parameter $q > 1$, and it vanishes entirely in the limiting (the classical) case.

Both the negative semiboundedness to the left of α_- and the additional self-adjointness to the left of α_{--} disappear in the classical case. Thus, it can be argued that the q-case exhibits a more complex behavior than the classical case.

Acknowledgments

The authors thank the three referees for their careful reading of this manuscript and their constructive comments.

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